Wave resistance to vertical motion in a stratified fluid

By F. W. G. WARREN
Imperial College, London

(Received 14 March 1959, and in revised form 12 July 1959)

When a body moves through a stratified fluid, i.e. one whose density decreases upwards, gravity waves are set up and this causes a resistance to motion. An axisymmetric case is considered in which a body moves steadily and vertically through a fluid whose density decreases exponentially upwards. The fluid is supposed perfect, incompressible, and unbounded in all directions. The equations of motion are linearized, and with a fairly general initial motion of the surrounding fluid, the limit of the solution as $t \to \infty$ is evaluated. Transform methods are used to solve the equation of motion, and the methods of steepest descents and stationary phase are used to obtain approximate solutions.

Streamlines and the distortion of the constant density levels for a spindle-shaped body are shown. The curves of resistance against a function of the velocity for the circular cylinder, the sphere, and a spindle-shaped body are also given. A criterion is given for when the maximum wave resistance for a sphere may be expected, and an estimate of this maximum resistance is made.

1. Introduction

The problem of a body rising through a stratified fluid has applications in the study of buoyant convection currents rising through stable surroundings. This type of motion frequently occurs in the atmosphere when a thermal rises through a cumulus cloud. In this case the expansion of the thermal during its ascent causes a condensation of the water vapour within it. The release of latent heat warms the thermal, and this causes a further expansion and a further decrease in density of the rising air mass. This offsets the loss in buoyancy as the thermal rises to higher, less dense regions, and the upward motion is maintained (Scorer 1958). For the purposes of this problem, the atmosphere may be regarded as an incompressible fluid whose density decreases upwards, and whose static stability parameter is the same as that of the atmosphere (Scorer 1950, 1958); and the thermal can be taken to have a fixed, spherical shape. What is required is a measure of the energy used to generate the gravity waves set up in the stable surroundings of the cumulus cloud as the thermal rises, so that the importance of this effect on the motion and growth of the thermal can be estimated. This is obtained in § 10.

The problem considered in this paper may also be regarded as a first approach to the study of the dispersion of stirring motions into gravity waves. A criterion is suggested for when the maximum amount of energy is used to make waves, and hence for when the maximum amount of energy is absorbed and dispersed by gravity waves.

2. The equation of motion

Perturbation methods are applied to the basic equations of flow to obtain a linear equation governing the stream function of the perturbation.

The notation employed, is, using, cylindrical co-ordinates:

z = height (the z-axis is coincident with that of the body),

r = horizontal co-ordinate,

(u, w) = velocity of the fluid,

W = steady vertical velocity of the body (positive upwards),

β = static stability parameter $= -\dfrac{1}{\rho_0}\dfrac{d\rho_0}{dz}$ = constant > 0,

where $\rho_0(z)$ = undisturbed density,

$$\alpha^2 = g\beta,$$

and

$$k_0 = \left|\frac{\alpha}{W}\right|.$$

If we write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + (w - W)\frac{\partial}{\partial z},$$

the equations of motion relative to axes fixed in the body are

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{1}$$

and

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g. \tag{2}$$

Since the fluid is incompressible

$$\frac{D\rho}{Dt} = 0, \tag{3}$$

and the equation of continuity reduces to

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0. \tag{4}$$

Equation (4) implies the existence of a stream function, $\psi(r, z, t)$, such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}.$$

We now write $\rho = \rho_0 + \rho'$ and $p = p_0 + p'$, where ρ' and p' are small perturbations of the undisturbed density and pressure, ρ_0 and p_0, respectively. If we substitute $\rho_0 + \rho'$ for ρ, and $p_0 + p'$ for p in equations (1), (2) and (3) and neglect products of u, w, p', ρ' and their derivatives, these three equations become

$$\rho_0\left(\frac{\partial}{\partial t} - W\frac{\partial}{\partial z}\right)u + \frac{\partial p'}{\partial r} = 0, \tag{1$'$}$$

$$\rho_0\left(\frac{\partial}{\partial t} - W\frac{\partial}{\partial z}\right)w + \frac{\partial p'}{\partial z} = -\rho' g \tag{2$'$}$$

and

$$\left(\frac{\partial}{\partial t} - W\frac{\partial}{\partial z}\right)\rho' + w\frac{\partial \rho_0}{\partial z} = 0. \tag{3$'$}$$

The elimination of ρ' and p' from these equations and the substitution for u and w in terms of ψ leads to the following equation of motion:

$$\left(\frac{\partial}{\partial t}-W\frac{\partial}{\partial z}\right)^2\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)-\frac{\beta}{r}\frac{\partial\psi}{\partial z}\right\}+\alpha^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0. \tag{5}$$

The first two terms in the curly brackets represent the vorticity, the term containing β is an inertia term which arises because the density decreases upwards, and the last term represents the effect of gravity.

If it assumed that a steady motion exists for which $\partial/\partial t \equiv 0$, then equation (5) becomes

$$\frac{\partial^2}{\partial z^2}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)-\frac{\beta}{r}\frac{\partial\psi}{\partial z}\right\}+k_0^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0. \tag{5$'$}$$

Equation (5)$'$ has the same form for both W and $-W$. Thus the equation of steady motion makes no distinction between ascent and descent at equal speeds. The assumption that there is no perturbation of the fluid at large distances from the body, together with the boundary condition at the surface of the body, leads to a solution of equation (5)$'$ in which a wave pattern appears below the body. This seems unreasonable if the body is in descent. On the other hand, if the assumption of no perturbation at infinity is dropped, the solution of equation (5)$'$ is indeterminate. In some cases it may be argued that the term containing β may be neglected because it is small compared with the other terms of equations (5) and (5)$'$. These equations then become

$$\left(\frac{\partial}{\partial t}-W\frac{\partial}{\partial z}\right)^2\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right\}+\alpha^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0 \tag{6}$$

and

$$\frac{\partial^2}{\partial z^2}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right\}+k_0^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0 \tag{6$'$}$$

respectively. The same equations are obtained if β and g are regarded as variable parameters, and $\beta \to 0$, $g \to \infty$, in such a way that α^2 remains constant.

However, the solution of the equation of steady motion (6)$'$ is indeterminate, even if the assumption of no perturbation at infinity is made. Also the circumstances in which the term in equation (5) containing β may be neglected do not seem obvious. For these reasons it is better to use the equation of unsteady motion (5), and with a general type of initial motion to look for a limiting value of the solution as $t \to \infty$.

3. The boundary conditions

If line sources and sinks are placed on the z-axis, the separating streamlines may be regarded, in the usual manner, as the outline of a body. The strength of these line sources may be written conveniently as

$$-2\pi W f'(z) f(z),$$

the accent denoting differentiation with respect to z. At points near the axis,

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z}=-\frac{W}{r}f'(z)f(z),$$

and hence

$$\psi(0,z,t)=-\frac{W}{2}\{f(z)\}^2=\psi_a, \quad \text{say,} \tag{7}$$

where ψ is defined so that the constant of integration is zero. If the axisymmetric body whose shape is given by $r_a = f(z)$ is a slender one, so that r_a is small, then

$$\frac{\partial}{\partial z}\psi(r_a, z, t) \simeq \frac{\partial}{\partial z}\psi(0, z, t),$$

and
$$\frac{u(r_a, z, t)}{W-w} \simeq \frac{1}{Wf(z)}\frac{\partial}{\partial z}\psi(0, z, t) \quad \text{if} \quad |w| \ll |W|.$$

However, if $r_a = f(z)$ is a boundary of the fluid, then

$$-\frac{u(r_a, z, t)}{W-w} = f'(z).$$

Hence equation (7) holds for a body whose shape is given approximately by $r_a = f(z)$, if the body is a slender one.

The flow pattern at $t = 0$ is given (compatibly) as

$$\left.\begin{aligned} \psi(r, z, 0) &= \psi_0 \\ \text{and} \qquad \frac{\partial}{\partial t}\psi(r, z, 0) &= \dot{\psi}_0. \end{aligned}\right\} \tag{8}$$

The initial motion described by ψ_0 and $\dot{\psi}_0$ is taken to be of a general nature and may contain waves. Alternatively, the body may be started suddenly from rest in a still fluid. The solution of the equation of impulsive motion (i.e. equation (5)$'$ with $g = 0$) is equivalent to a potential flow solution multiplied by a factor $\cos\frac{1}{2}\beta r$. Thus the motion in the horizontal direction has a 'wavelength' of $2/\beta$; but this seems of theoretical interest only since the amplitude of the motion dies away rapidly in an exponential manner.

4. The transformation of the equation of motion

Equation (5) is transformed by means of a Fourier transform in the vertical direction and a Hankel-like transform in the horizontal direction:

$$\Psi(r, k, t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} dz\, e^{ikz}\,\psi(r, z, t), \tag{9}$$

$$\chi(m, k, t) = \int_0^{\infty} dr\, J_1(mr)\,\Psi(r, k, t). \tag{10}$$

The transforms corresponding to ψ_0, $\dot{\psi}_0$ and ψ_a are

$$\Psi_0, \chi_0;\ \dot{\Psi}_0, \dot{\chi}_0 \quad \text{and} \quad \Psi_a(k) = \Psi(0, k, t)$$

respectively. If we multiply equation (5) by e^{ikz} and integrate with respect to z from $-\infty$ to $+\infty$, and assume ψ vanishes at $\pm\infty$, we obtain

$$\left(\frac{\partial}{\partial t} + iWk\right)^2 \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) - k^2\frac{\Psi}{r} + i\beta k\frac{\Psi}{r}\right\} + \alpha^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = 0.$$

If this equation is multiplied by $rJ_1(mr)$ and integrated with respect to r from 0 to ∞, it becomes

$$\left[\left(\frac{\partial}{\partial t} + iWk\right)^2 (k^2 - i\beta k + m^2) + \alpha^2 m^2\right]\chi = m(\alpha^2 - W^2k^2)\,\Psi_a(k). \tag{11}$$

To derive equation (11), it has been assumed that $J_1(mr)\,(\partial/\partial r)\,\Psi(r,k,t)$ vanishes at $r = 0$ and $r = \infty$ and that $\Psi(r,k,t)$ vanishes at $r = \infty$; it then follows that

$$\int_0^\infty dr\, rJ_1(mr)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = m\Psi(0,k,t) - m^2\chi(m,k,t).$$

Equation (11) is an ordinary differential equation with t as the independent variable. Its solution may be written as the sum of a particular integral and a complementary function: $\chi = \chi_1 + \chi_2$. Correspondingly, $\Psi = \Psi_1 + \Psi_2$ and $\psi = \psi_1 + \psi_2$.

5. The particular integral

The particular integral of equation (11) is obtained by putting $\partial/\partial t \equiv 0$, since the right-hand side is independent of time. This gives $\chi_1 = m\Psi_a/(m^2-\lambda^2)$, where

$$\lambda^2 = \frac{k^4 - i\beta k^3}{k_0^2 - k^2}.$$

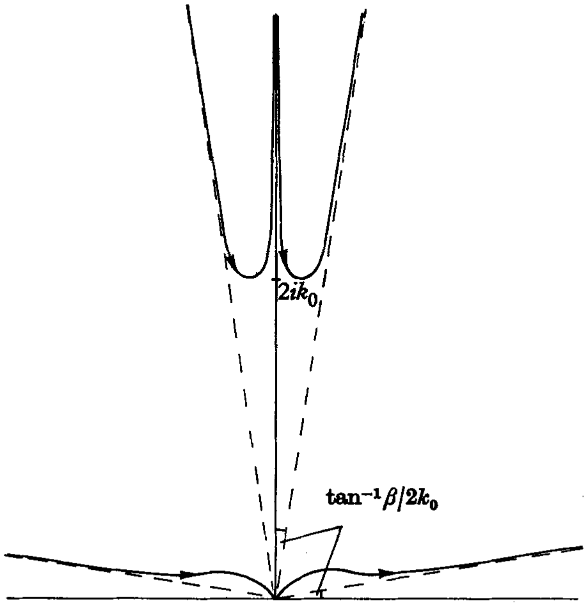

FIGURE 1. Path for λ in the complex λ-plane as k varies.

The inverse of the transformation (10) is

$$\Psi(r,k,t) = r\int_0^\infty dm\, mJ_1(rm)\,\chi(m,k,t).$$

Hence,
$$\Psi_1(r,k,t) = r\Psi_a(k)\int_0^\infty dm\,\frac{m^2J_1(rm)}{m^2-\lambda^2}.$$

This integral is evaluated in Watson (1944, p. 424).† Substituting its value, we obtain

$$\Psi_1(r,k,t) = (\tfrac{1}{2}\pi i)\, r\lambda H_1^{(1)}(r\lambda)\,\Psi_a(k),$$

where
$$\operatorname{Im}(\lambda) > 0.$$

† The integral $\displaystyle\int\frac{z^2H_1^{(1)}(rz)\,dz}{z^2-\lambda^2}$ taken round an infinite semicircle in the upper half-plane gives the results immediately.

The inverse of the transform (9) then gives

$$\psi_1(r,z,t) = \frac{\pi i}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{-ikz} r\lambda H_1^{(1)}(r\lambda)\, \Psi_a(k),$$

where $\mathrm{Im}\,(\lambda) > 0$. The path of λ, as k varies from $-\infty$ to $+\infty$, is shown in figure 1. If $\beta/k_0 \ll 1$, this path lies approximately on the real and imaginary axes. If it is assumed that the body is symmetrical about its centre so that $r_a = f(z) = f(-z)$, then $\Psi_a(k)$ is an even function of k. ψ_1 may then be written

$$\psi_1 = -\sqrt{\frac{\pi}{2}}\,\mathrm{Im} \int_0^{\infty} dk\, \Psi_a(k)\, e^{ikz} r\Lambda H_1^{(1)}(r\Lambda), \tag{12}$$

where

$$\Lambda = \begin{cases} \lambda_1 & \text{if} \quad k < k_0, \\ i\lambda_2 & \text{if} \quad k > k_0 \end{cases}$$

and

$$\lambda_1^2 = \frac{k^4}{k_0^2 - k^2} \quad (\lambda_1 > 0),$$

$$\lambda_2^2 = \frac{k^4}{k^2 - k_0^2} \quad (\lambda_2 > 0).$$

6. The complementary function

Since the particular integral is independent of time, the behaviour of ψ as $t \to \infty$ depends solely upon the behaviour of ψ_2 as $t \to \infty$. The complementary function is obtained if we write $\Psi_a(k) = 0$ in equation (11). This gives

$$\chi_2 = A_1(k,m)\, e^{\gamma_1 t} + A_2(k,m)\, e^{\gamma_2 t},$$

where A_1 and A_2 are functions of k and m, and γ_1 and γ_2 are the roots of

$$(\gamma + iWk)^2\,(k^2 - i\beta k + m^2) + \alpha^2 m^2 = 0. \tag{13}$$

At $t = 0$, χ and $\partial\chi/\partial t$ have the given values χ_0 and $\dot{\chi}_0$, respectively, and A_1 and A_2 are found from these conditions. We thus obtain

$$A_1 = [\gamma_2(\chi_0 - \chi_1) - \dot{\chi}_0]/[\gamma_2 - \gamma_1],$$
$$A_2 = [\gamma_1(\chi_0 - \chi_1) - \dot{\chi}_0]/[\gamma_1 - \gamma_2].$$

The inverse of the transforms (8) and (9) gives

$$\psi_2 = \frac{r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \int_0^{\infty} dm\, e^{-ikz} m J_1(rm)\, [A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t}].$$

A transformation to polar co-ordinates is now made, so that one range of integration is finite. We write $k = \rho \sin\theta$, $m = \rho\cos\theta$ and thus obtain

$$\psi_2 = \frac{r}{\sqrt{2\pi}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_0^{\infty} d\rho\, \rho^2 \cos\theta \exp[-iz\rho\sin\theta] \times J_1(r\rho\cos\theta)\,[A_1(\rho,\theta)\, e^{\gamma_1 t} + A_2(\rho,\theta)\, e^{\gamma_2 t}],$$

where

$$\gamma_{1,2} = -iW\rho\sin\theta \pm i\alpha\cos\theta \sqrt{\frac{\rho}{\rho - i\beta\sin\theta}}.$$

We write

$$I_1^{(+)} = \frac{r}{\sqrt{2\pi}} \int_0^{\frac{1}{2}\pi} d\theta \int_0^\infty d\rho\, \rho^2 \cos\theta \exp\left[-iz\rho\sin\theta\right] J_1(r\rho\cos\theta)\, A_1(\rho,\theta)\, e^{\gamma_1 t},$$

$$I_1^{(-)} = \frac{r}{\sqrt{2\pi}} \int_{-\frac{1}{2}\pi}^{0} d\theta \int_0^\infty d\rho\, \rho^2 \cos\theta \exp\left[-iz\rho\sin\theta\right] J_1(r\rho\cos\theta)\, A_1(\rho,\theta)\, e^{\gamma_1 t}.$$

Similarly the notations $I_2^{(+)}$ and $I_2^{(-)}$ are used for integrals containing A_2.

To find the limit of the integrals as $t \to \infty$, we consider the ρ-integration first, and deform the paths of integration in the complex ρ-plane to regions where $\mathrm{Re}\,(\gamma_{1,2}) < 0$.† Two cases arise, corresponding to ascent and descent of the body. For ascent, it is found that if certain assumptions about the initial conditions are made, no poles are crossed in the deformation in the ρ-plane. The whole path may be deformed to regions where $\mathrm{Re}\,(\gamma_{1,2}) < 0$. Hence in this case the integrals tend to zero as $t \to \infty$. For the case of descent, the same assumptions are made, but poles of χ_1 are crossed at points where $\gamma = 0$, and the residues give rise to a steady term as $t \to \infty$. A difficulty arises for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ because it is not possible to deform the whole of the path in the ρ-plane to regions where

$$\mathrm{Re}\,(\gamma_{1,2}) < 0.$$

The second integration with respect to θ then shows that these integrals do not have limits as $t \to \infty$. However, if the assumption is made that $\beta/k_0 \ll 1$ this difficulty is partly overcome in the sense that the integrals $I_1^{(-)}$ and $I_2^{(+)}$ taken along the deformed paths remain negligible for a certain interval of time. During this time the integrals are approximately equal to the residue terms from the poles crossed by the deformations. From these residues the limit of ψ_2 is obtained. Thus it may be shown that

$$\lim_{t\to\infty} \psi_2 = \begin{Bmatrix} 0, & \text{if} \quad W > 0 \\ \sqrt{2\pi}\displaystyle\int_0^{k_0} dk\, \Psi_a(k)\, r\Lambda J_1(r\Lambda)\sin kz & \text{if} \quad W < 0. \end{Bmatrix} \tag{14}$$

The remainder of this section gives a proof of the expression (14) and discusses some points which arise from the preceding paragraph.

Case I. Ascent $W > 0$

It is assumed that A_1 and A_2 are $O(\exp[b|\mathrm{Im}\,\rho|]\rho^{-3-\Delta})$ as $|\rho| \to \infty$, where $\Delta > 0$ and $b \geqslant 0$. Equation (13) shows that if $\gamma = 0$, then $m^2 = \lambda^2$. Hence χ_1 has poles at points where $\gamma = 0$, A_1 has a pole at $\gamma_1 = 0$ and A_2 has a pole at $\gamma_2 = 0$. Figure 2 shows the ρ-plane. Regions where $\mathrm{Re}\,(\gamma) < 0$ are shaded, and the poles of $A_{1,2}$ at $\gamma_{1,2} = 0$ are shown at the points marked Γ. γ has a pole at $\rho = i\beta\sin\theta$ and a branch point at the origin. A cut is made along the imaginary ρ-axis to join these points, so that γ is single valued in the cut plane.

We consider the upper bounds of the integrands of the integrals $I_1^{(+)}$, etc., taken along the deformed paths. Upper bounds for the integrals taken along the deformed paths may then be found. The range of the θ-integration is divided into the ranges $(0, \pm\epsilon)$ and $(\pm\epsilon, \pm\frac{1}{2}\pi)$, where ϵ is a small positive number.

† The transformation $\rho \to \rho(\gamma)$ is not a simple one and it is better to keep to the independent variables (ρ, θ).

Discussion of the integral $I_1^{(+)}$

(i) No deformation is made in the interval $0 < \theta < \epsilon$, and it may be shown that $\left|\int_0^\infty d\rho\right| < M$, for some fixed M.

(ii) In $\epsilon < \theta < \frac{1}{2}\pi$, the path of integration in the ρ-plane is deformed as shown in figure 3.

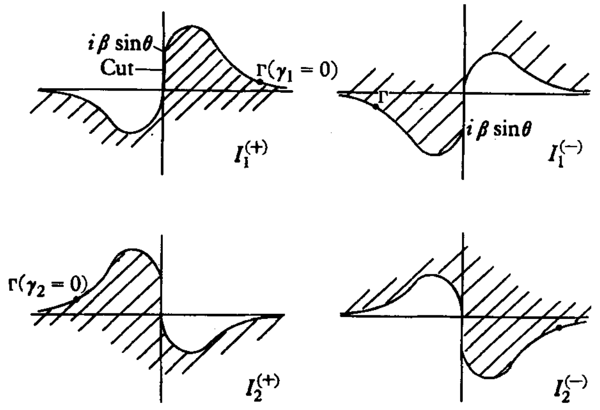

FIGURE 2. Case I. Paths of Re $(\gamma) = 0$ in the complex ρ-plane.

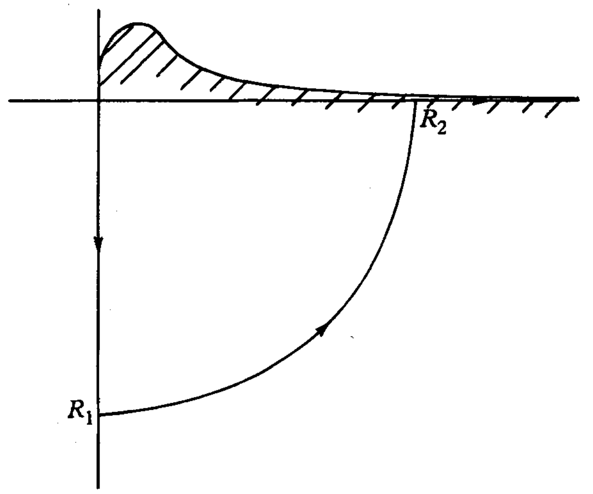

FIGURE 3. Case I. Deformation for $I_1^{(+)}$ in $\epsilon < \theta < \frac{1}{2}\pi$.

It follows that

$$\left|\int_{R_1 R_2 \infty} d\rho\right| < K\epsilon$$

by suitable choice of R_1, and that

$$\left|\int_{0R_1} d\rho\right| < \frac{K}{Wt\sin\theta - 2(r + |z| + b)}$$

if $Wt\sin\theta > 2(r + |z| + b)$, where K is a fixed number.

Discussion of the integral $I_1^{(-)}$.

(i) In the interval $-\epsilon < \theta < 0$ the path is deformed as shown in figure 4.

It may be shown that $\left|\int_{0,\infty} d\rho\right| < M\, e^{ck_0(r+|z|+b)}$, where c is some constant which depends on β and k_0 only.

(ii) In $-\frac{1}{2}\pi < \theta < -\epsilon$, the deformation is similar to that used in $I_1^{(+)}$, see figure 5.

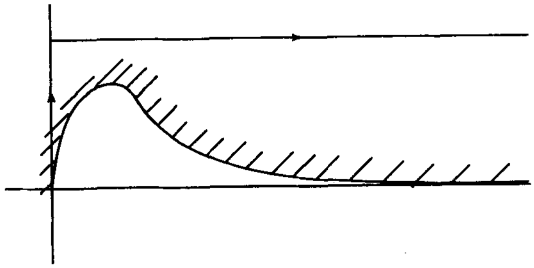

FIGURE 4. Case I. Deformation for $I_1^{(-)}$ in $-\epsilon < \theta < 0$.

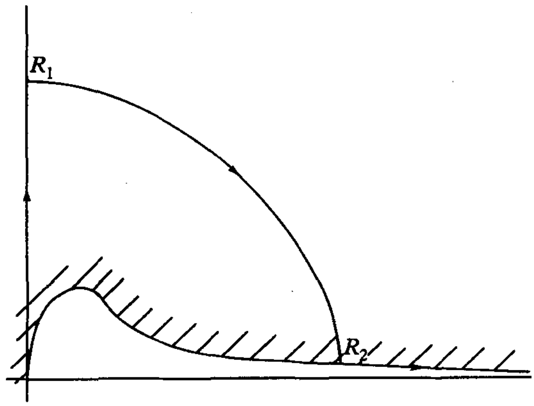

FIGURE 5. Case I. Deformation for $I_1^{(-)}$ in $-\frac{1}{2}\pi < \theta < -\epsilon$.

In this case it may be shown that

$$\left|\int_{0R_1R_2\infty} d\rho\right| < K\epsilon + \frac{K}{Wt\sin\theta - 2(r+|z|+b)}.$$

Similar results hold for the integrals $I_2^{(+)}$ and $I_2^{(-)}$, when the paths are deformed in a similar manner. If we collect these results for the integrals $I_1^{(+)}$, etc., and consider upper bounds for varying θ, we obtain the following upper limit for the θ-integration:

$$\left|\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho\right| < K\left\{(1+\exp\left[ck_0(r+|z|+b)\right])\,\epsilon + \frac{1}{Wt_0\sin\epsilon - 2(r+|z|+b)}\right\},$$

where $Wt_0 \sin\epsilon > 2(r+|z|+b)$.

For poles crossed by the deformations there will be contributions from the residues, which give terms like

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \Theta\, e^{\gamma_p t}. \tag{15}$$

Here Θ is some function of θ and a pole occurs at $\rho = \rho_p(\theta)$, say, where

$$\gamma = \gamma(\rho_p, \theta) = \gamma_p(\theta).$$

If $\mathrm{Re}(\gamma_p) < 0$, the expression (15) will decrease with time, but if $\mathrm{Re}(\gamma_p) > 0$ (e.g. for the deformation used for $I_2^{(-)}$), this may not be so. In general, if it is possible to deform the path of integration in the θ-plane to regions where $\mathrm{Re}(\gamma_p) < 0$ (where γ_p is such that $\mathrm{Re}(\gamma_p(\pm\frac{1}{2}\pi)) \leqslant 0$), the expression (15) will decrease with time. However, this deformation is not always possible, and, for example, we may have $\mathrm{Im}(\gamma_p) = \text{constant} = Wl$, say, and $\mathrm{Re}(\gamma_p) = 0$. There is then a steady fluctuation present throughout the motion. This type of motion is easier to see if we use equation (6). The assumption that

$$\psi \propto e^{-ikz} r J_1(mr)\, e^{iWlt}$$

then gives

$$(l+k)^2(k^2+m^2) = k_0^2 m^2. \tag{13'}$$

Hence equation (6) has a solution

$$\psi = r\,e^{iWlt} \int_0^\infty dk\, \Psi_a(k)\, e^{-ikz} m J_1(rm),$$

where m satisfies equation (13′) and l is fixed.

This fluctuating type of motion does not seem to be of practical interest since it must be present in the initial conditions and it is not easy to see how this motion could arise in the first instance, unless a fluctuating type of disturbance were present.

With regard to the θ-integration for the residues in the case where

$$\mathrm{Im}(\gamma_p) = \text{constant}, \quad \mathrm{Re}(\gamma_p) > 0,$$

the fluctuation would increase with time. In this case the linear theory is inadequate to say what eventually happens.

If it is now assumed that poles of the above description do not occur in A_1 and A_2 and hence in χ_0 and $\dot{\chi}_0$, i.e. that certain types of waves are absent from the initial condition, then

$$|\psi_2| < K\left\{(1+\exp[ck_0(r+|z|+b)])\,\epsilon + \frac{1}{Wt\sin\epsilon - 2(r+|z|+b)}\right\}.$$

This holds for all (r_1, z_1, t) if $r_1 + |z_1| < r + |z|$ and $t > t_0$. Hence $\psi_2 \to 0$ as $t \to \infty$ if $W > 0$ for all regions of the flow. This result holds for any $\beta > 0$.

Case II. Descent $W < 0$

The same methods are used as in Case I, but it is no longer possible to deform the paths of integrations for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ in the ρ-plane so that they lie completely in regions where $\mathrm{Re}(\gamma) < 0$. The singularities of γ prevent this. Figure 6, which corresponds to figure 2 of Case I, shows the ρ-plane.

The paths for the integrals $I_1^{(-)}$ and $I_2^{(+)}$ are deformed as shown in figures 8, 9 and 10. In the regions were $\mathrm{Re}(\gamma) > 0$, the paths are deformed so as to pass through a col as in the method of steepest descent. An upper bound for the contributions from segments of the paths where $\mathrm{Re}(\gamma) > 0$ is then made. The track of the cols for varying θ is shown in figure 7.

The position of the cols is given by $\rho = \rho_c(\theta)$, say, and at these points

$$\gamma = \gamma(\rho_c, \theta) = \gamma_c(\theta).$$

We write
$$\operatorname{cosec}\theta^* \cot\theta^* = \frac{3\sqrt{3}}{8}\frac{\beta}{k_0} \quad (\tfrac{1}{2}\pi > \theta^* > 0)$$

and
$$\operatorname{cosec}\theta^{**} \cot\theta^{**} = \frac{1}{2}\frac{\beta}{k_0} \quad (\tfrac{1}{2}\pi > \theta^{**} > 0).$$

Then if $|\theta| > \theta^*$, it may be shown that two cols lie on the cut; and that if $|\theta| > \theta^{**}$ both zeros of γ lie on the cut. We suppose that β and g are variable parameters

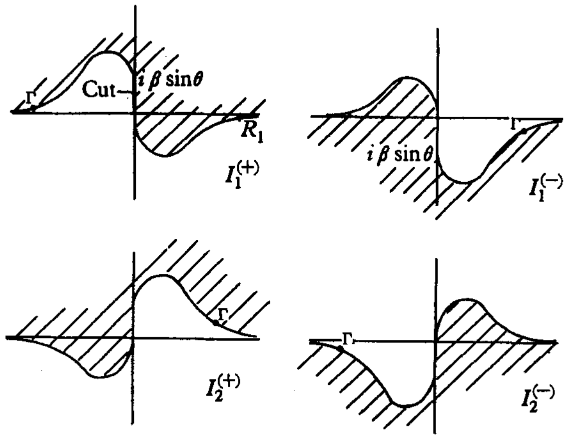

FIGURE 6. Case II. Paths of $\mathrm{Re}(\gamma) = 0$ in the complex ρ-plane.

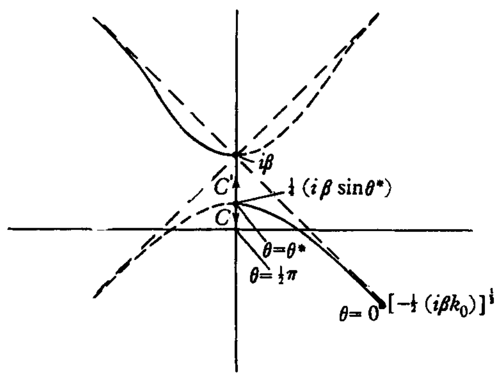

FIGURE 7. Case II. Path of the cols for varying θ in the complex ρ-plane.

with α, k_0 and W held fixed, and that $0 < \beta/k_0 \ll 1$. The range of the θ-integration is divided into the ranges $(0, \pm\epsilon)$; $(\pm\epsilon \pm \tfrac{1}{2}\pi \mp \delta)$; and $(\pm\tfrac{1}{2}\pi \mp \delta, \pm\tfrac{1}{2}\pi)$; where ϵ and δ are small positive numbers. β is chosen so that $\tfrac{1}{2}\pi - \theta^* < \delta$ in order to restrict the contributions from the paths in regions where $\mathrm{Re}(\gamma) > 0$.

Discussion of the integral $I_1^{(+)}$

(i) There is no deformation of the path in $0 < \theta < \epsilon$ and it may be shown that

$$\left|\int_0^\infty d\rho\right| < M.$$

(ii) In $\epsilon < \theta < \frac{1}{2}\pi$, the path is deformed to $\mathrm{Re}(\gamma) = 0$. On this path it may be shown that

$$\left|\frac{d\gamma/d\rho}{W\sin\theta}\right| > 1.$$

With the assumption that $A(\rho, \theta)$ is of bounded variation on any finite length of the path, it follows that

$$\left|\int_{0,\infty} d\rho\right| < \frac{K\exp[ck_0(r+|z|+b)]}{Wt\sin\theta} + M\epsilon.$$

Similar results hold for $I_2^{(-)}$.

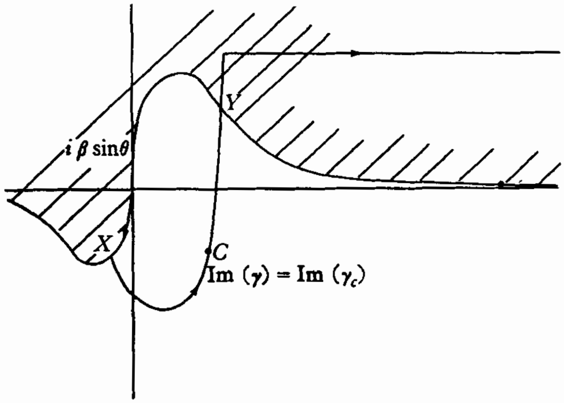

FIGURE 8. Case II. Deformation for $I_2^{(+)}$ in $0 < \theta < \epsilon$.

Discussion of the integral $I_2^{(+)}$

(*a*) *Contributions from the paths which lie in regions where* $\mathrm{Re}(\gamma) \leqslant 0$

(i) In $0 < \theta < \epsilon$, the path is deformed as shown in figure 8.

It may be shown that

$$\left|\int_{OX+Y\infty} d\rho\right| < M\exp[ck_0(r+|z|+b)].$$

(ii) In $\epsilon < \theta < \frac{1}{2}\pi - \delta$, the path is deformed as in figure 9. Here

$$\left|\int_{OX} d\rho\right| < \frac{K\exp[ck_0(r+|z|+b)]}{Wt\sin\theta},$$

as may be shown by a discussion similar to that used above for the integral $I_1^{(+)}$, (ii).

If $\beta/k_0 \ll \sin\delta$, it may be shown that

$$\left|\frac{d\gamma/d\rho}{W\sin\theta}\right| > \frac{1}{2}$$

and that the integrand is uniformly bounded with respect to β and θ on the deformed path.

It then follows that

$$\left|\int_{YZ} d\rho\right| < \frac{K\exp[ck_0(r+|z|+b)]}{Wt\sin\theta},$$

where $|OZ| = k_0$, say.

As in Case I, integral $I_1^{(+)}$, it may also be shown that

$$\left|\int_{Z\infty} d\rho\right| < \frac{K}{Wt\sin\theta - 2(r+|z|+b)} + M\epsilon.$$

(iii) In $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$, the paths are deformed in the three subintervals

$$\tfrac{1}{2}\pi - \delta < \theta < \theta^*; \quad \theta^* < \theta < \theta^{**}; \quad \theta^{**} < \theta < \tfrac{1}{2}\pi;$$

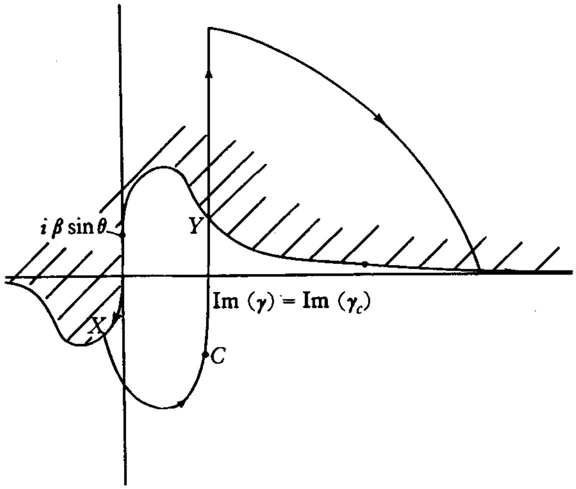

FIGURE 9. Case II. Deformation for $I_2^{(+)}$ in $\epsilon < \theta < \frac{1}{2}\pi$.

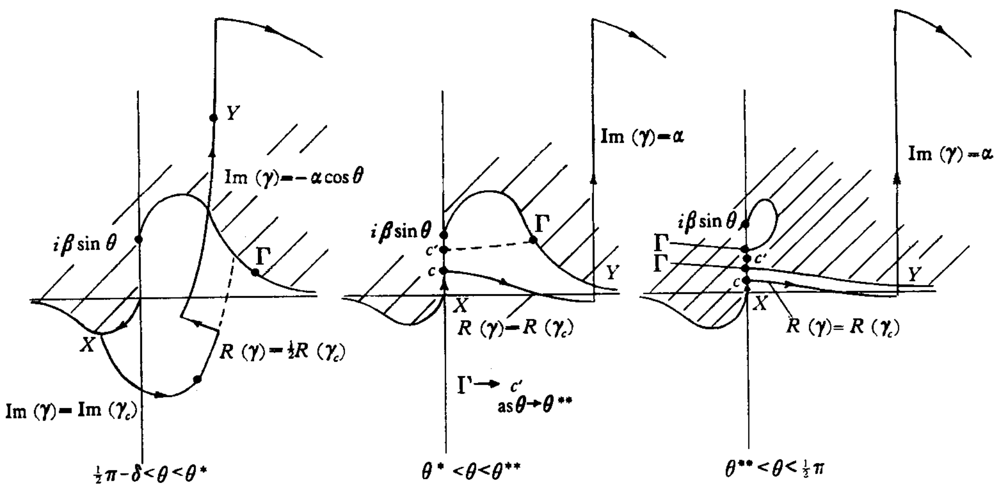

FIGURE 10. Case II. Various deformations for $I_2^{(+)}$ in $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$.

as shown in figure 10. For points on the segment $Y\infty$ of the deformed path, where $|OY| = k_0$, it may be shown that

$$\left|\frac{d\gamma/d\rho}{W\sin\theta}\right| > \frac{1}{2}$$

Similar arguments hold for $\left|\int_{Y\infty} d\rho\right|$ as in (ii) above.

(*b*) *Contributions from the paths which lie in regions where* $\mathrm{Re}(\gamma) \geqslant 0$

If we consider the path of the col, we may show that in the interval

$$0 < \theta < \theta^*, \quad \mathrm{Re}(\gamma_c/W) < 5\sqrt{(\beta k_0)};$$

and that in

$$\theta^* < \theta < \tfrac{1}{2}\pi, \quad \mathrm{Re}(\gamma_c/W) < \tfrac{1}{8}\beta.$$

(i) In $0 < \theta < \frac{1}{2}\pi - \delta$, it may be shown that the length of the path XY is less than

$$K\sqrt{(\beta k_0)}.$$

(ii) In $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi$, the length of the path in the region where

$$\mathrm{Re}(\gamma) \geqslant 0 \quad \text{is} \quad O(k_0).$$

It may also be shown that the integrand is uniformly bounded with respect to β and θ on the deformed paths. From these results it may then be shown that

$$\left|\int_0^{\frac{1}{2}\pi} d\theta \int_{XY} d\rho\right| < K[\sqrt{(\beta k_0)} + \delta] \exp[ck_0(r+|z|+b)] \exp[5W\sqrt{(\beta k_0)}\, t].$$

The integral $I_1^{(-)}$ may be discussed in a manner similar to $I_2^{(+)}$.

From the above results for the integrals $I_1^{(+)}$, etc., we obtain the following upper limit for the θ-integration:

$$\left|\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho\right| < K\exp[ck_0(r+|z|+b)]\left\{\epsilon + \frac{1}{Wt\sin\epsilon} + \frac{1}{Wt\sin\epsilon - 2(r+|z|+b)} + [\sqrt{(\beta k_0)} + \delta]\exp[5W\sqrt{(\beta k_0)}\, t]\right\}.$$

This holds for all $0 < \beta < \bar{\beta}$, for some $\bar{\beta} > 0$. (c does not increase if β decreases.) It also holds for all (r_1, z_1, t), where $r_1 + |z_1| < r + |z|$, and $t > t_0$.

If we choose $t_1 \gg t_0$ and β such that $\sqrt{(\beta k_0)} \ll 1/5Wt_1$ (i.e. $\sqrt{(\beta/k_0)} \ll 1/5\alpha t_1$), then

$$\left|\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho\right| < K\exp[ck_0(r+|z|+b)](3\epsilon + 4\delta) \tag{16}$$

during some interval of time (t_1, t_2), $t_2 > t_1$. If (ϵ, δ) is chosen so that the expression on the right-hand side of (16) is small compared with the average value of

$$|\psi_1 + \text{residue terms}|\dagger$$

in the region $r_1 + |z_1| < r + |z|$, then

$$\left|\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{\text{paths}} d\rho\right|$$

may be neglected in the interval $t_1 < t < t_2$. This interval may be called a 'quasi-limit' interval. An estimate of $\psi_2 = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_0^\infty d\rho$ for large t may be found from the asymptotic expression of $\int_{XY} d\rho$. The method of steepest descent through the col gives the first n-terms in the form‡

$$\sum_{r=1}^{n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \Theta_r \exp[\gamma_c(\theta)\, t]\, t^{-r-\frac{1}{2}},$$

† ψ_1 and the residue terms change little as $\beta \to 0$.

‡ A modification is needed at $\theta = \theta^*$ since $d\gamma/d\rho = 0$ there.

where the Θ_r are functions of θ, and $\mathrm{Re}\,(\gamma_c(\pm\frac{1}{2}\pi)) = \mathrm{Re}\,(\gamma_c(0)) = 0$. A different form of deformation of the path in the ρ-plane occurs at $\theta = \theta^*$ and hence not all the Θ_r are holomorphic at this point. Hence complete deformation of the path of integration in the θ-plane to regions where $\mathrm{Re}\,(\gamma_c) \leqslant 0$ is not possible,† and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{XY} d\rho$$

gives terms

$$O\left(\frac{\exp\left[\frac{1}{8}(\beta\,|W|\,t)\right]}{t^{r+\frac{1}{2}}}\right) + \ldots,$$

where r is some integer. Hence $|\psi_2|$ diverges as $t \to \infty$, and eventually the motion increases without limit.

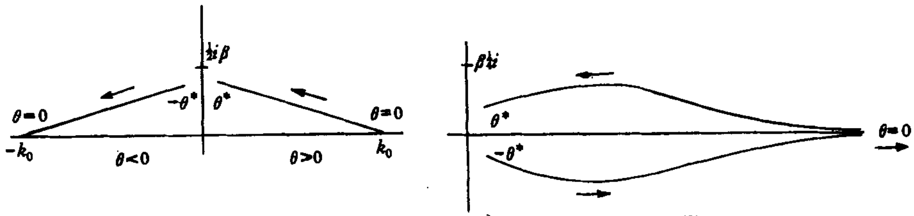

FIGURE 11. Paths of integration for k, (*a*), and λ, (*b*).

The assumption that second-order terms are small no longer holds at this stage and the linear theory seems inadequate. Physically, the medium is a dispersive one in which local disturbances are propagated outwards in all directions. The gradually increasing inertia forces retard the propagation to deeper levels, and waves ahead cannot disperse so rapidly in descent as in ascent.

Similar assumptions concerning the poles of χ_0 and $\dot{\chi}_0$ are made as in Case I. In the case where the motion is started suddenly from rest, a pole of χ_0 occurs at $\rho = i\beta \sin\theta$. It seems better to treat this pole separately, since the arguments used earlier would not hold in this case.

With the above assumptions, the limit of the complementary function in the 'quasi-limit' interval is found from the residues of the poles of χ_1 at $\gamma = 0$, which are crossed by the deformations used for $I_1^{(-)}$ and $I_2^{(+)}$. If we substitute for the residues, after a little manipulation we obtain

$$-\sqrt{\frac{\pi}{2}}\, ir \int_L dk\, \Upsilon_a(k)\, e^{-ikz}\, \lambda J_1(r\lambda)\, \mathrm{sgn}\,\theta,$$

where L is the path in the k-plane shown in figure 11. Figure 11 also shows the corresponding path for λ.

If $\beta r \ll 1$ and $\beta z \ll 1$, then these paths may be taken to lie approximately on the real axes. With the assumption that $\Upsilon_a(k)$ is even, this leads to the expression

$$\sqrt{2\pi} \int_0^{k_0} dk\, \Upsilon_a(k)\, r\Lambda J_1(r\Lambda) \sin kz. \tag{17}$$

† Other paths may be chosen, but a different form of deformation occurs at some stage because the contours of γ change their form at $\theta = \theta^*$ and at $\theta = \theta^{**}$.

In conclusion, for Case II, if $\beta/k_0 \ll 1$, there is a region around the body where, for a given time interval, ψ_2 is given by the expression (17). The length of this interval becomes infinite as $\beta/k_0 \to 0$.

7. The complete solution

The results of §§5 and 6 give the following expression for ψ as $t \to \infty$:

$$\psi = \frac{W}{2} \operatorname{Im} \int_0^\infty dk\, F(k)\, r\Lambda H_1^{(1)}(r\Lambda) \exp\left[i\,(\operatorname{sgn} W)\, kz\right], \tag{18}$$

where

$$F(k) = \int_0^\infty dz \cos kz \{f(z)\}^2.$$

Hence the wave pattern produced by a descending body is the same as that produced by an ascending one. The waves are downstream in each case. When the initial conditions are those of a steady, numerically greater velocity, and ψ_0 is given by expression (18), it may be shown that the same wave pattern eventually emerges.

Methods similar to those used in §§4–6, may be employed to solve equation (6). There is no essential singularity of the integrand in the corresponding double integral expression for ψ_2. The paths of $\operatorname{Re}(\gamma) = 0$ coincide with the real ρ-axis. The poles of χ_1 also occur on the real ρ-axis, and the double integral expressions for ψ_1 and ψ_2 are improper integrals. If principal values of the integrals are taken we obtain after one integration

$$\psi_1 = \frac{W}{2} \operatorname{Im} \int_0^\infty dk\, F(k)\, r\Lambda H_1^{(1)}(r\Lambda) \cos kz$$

and

$$\lim_{t\to\infty} \psi_2 = \frac{W}{2} \operatorname{Im} \int_0^{k_0} dk\, F(k)\, r\Lambda H_1^{(1)}(r\Lambda)\, i(\operatorname{sgn} W) \sin kz.$$

These two results differ from the corresponding expressions when $\beta \neq 0$. However, the sum $\psi = \psi_1 + \psi_2$ is the same in both cases, and in this sense the limit of the solution of equation (5) as $\beta \to 0$ is the same as the solution of the limiting form of equation (5) as $\beta \to 0$.

8. Asymptotic approximation to the motion

An asymptotic expression for the streamlines and the distortion of the constant density levels can be obtained by the method of steepest descents. The displacement of the streamlines is given approximately by

$$\Delta_s = \int_z^\infty dz \frac{u}{W} = -\frac{\psi}{Wr},$$

and the displacement Δ_{ρ_0} of the constant density levels by

$$\Delta_{\rho_0} = \int_z^\infty dz \frac{w}{W}.$$

If we write $k = k_0 \sin \xi$ and use the asymptotic expression for $H_1^{(1)}(x)$

$$H_1^{(1)}(x) \simeq \sqrt{\frac{2}{\pi x}} \exp\left[i(x - \tfrac{3}{4}\pi)\right],$$

then expression (18) yields

$$\Delta_s \simeq -\sqrt{\frac{k_0^3}{2\pi r}}\,\mathrm{Im}\int_M d\xi\, F(k_0\sin\xi)\sin\xi\cos^{\frac{1}{2}}\xi\exp\left[ik_0\sin\xi(z+r\tan\xi)-\tfrac{3}{4}\pi i\right],$$

when $W > 0$. M is the path shown in figure 12. It is deformed to pass through the col at C. This col contributes an exponentially small term to the asymptotic approximation to Δ_s and is omitted. It may be regarded as the potential flow displacement.

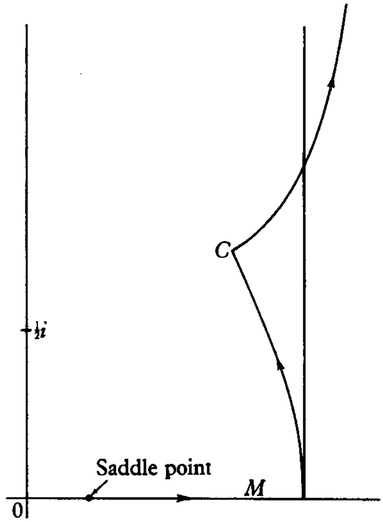

FIGURE 12. Path of integration for ξ.

When $z < 0$ a saddle point appears on the real axis between the origin and $(\frac{1}{2}\pi, 0)$. The method of stationary phase then gives

$$\Delta_s \simeq \frac{1}{r}\frac{k_0 F(k_0\sin\xi^*)}{\sqrt{(2+\sin^2\xi^*)}}\sin\xi^*\cos^2\xi^*\cos\left(\frac{zk_0\sin\xi^*}{1+\cos^2\xi^*}\right) \quad \text{if} \quad z<0$$

$$\simeq 0 \quad \text{if} \quad z > 0$$

for the asymptotic value of Δ_s. Here ξ^* satisfies the equation $dG/d\xi = 0$, where

$$G(\xi) = irk_0\sin\xi\left(\tan\xi+\frac{z}{r}\right),$$

and where $z/r = \tan\phi$, say. This gives

$$\tan^3\xi^* + 2\tan\xi^* + z/r = 0$$

or

$$\sin 2\xi^* + 2\tan(\xi^*-\phi) = 0.$$

A similar result holds for Δ_{ρ_0}:

$$\Delta_{\rho_0} \simeq -\frac{1}{r}\frac{k_0 F(k_0\sin\xi^*)}{\sqrt{(2+\sin^2\xi^*)}}\sin^2\xi^*\cos\xi^*\cos(rk_0\tan^2\xi^*\sec\xi^*) \quad \text{if} \quad z<0$$

$$\simeq 0 \quad \text{if} \quad z > 0.$$

If $W < 0$ the reverse holds with Δ_s and Δ_{ρ_0} asymptotically zero if $z < 0$.

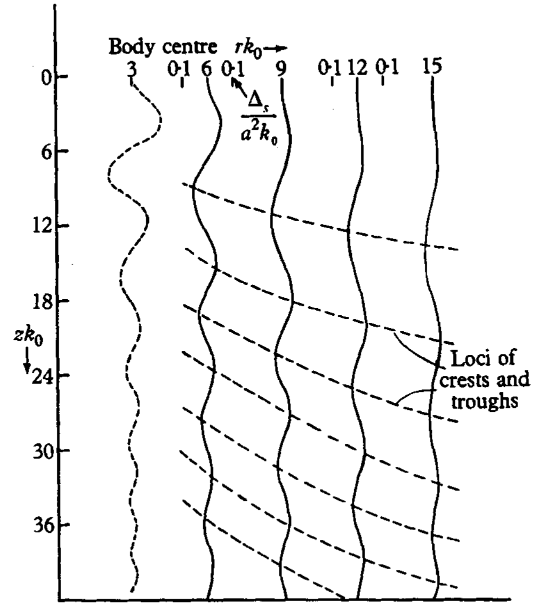

FIGURE 13. Streamlines for a spindle-shaped body.

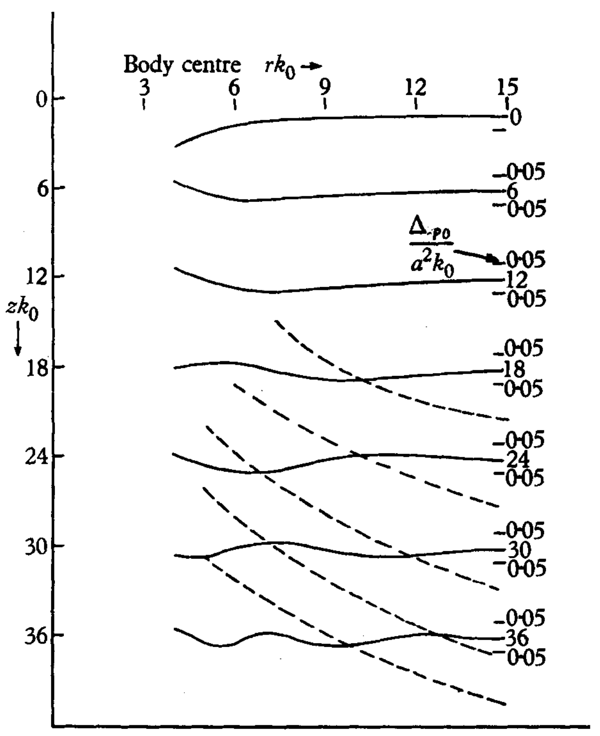

FIGURE 14. Deformation of the constant density levels for a spindle-shaped body.

If ξ^* is used as a parameter we may write the displacements as

$$\frac{1}{r} \times \text{amplitude factor} \times \cos\{(r \text{ or } z) \times \text{phase factor}\}.$$

The results for a spindle-shaped body whose shape is given by

$$\begin{aligned} f(z) &= a\cos\frac{\pi z}{2b} \quad \text{if} \quad |z| < b \\ &= 0 \qquad\qquad \text{if} \quad |z| > b \end{aligned}$$

have been plotted and are shown in the figures 13 and 14. a and b are the half-breadth and half-length of the body and k_0^{-1} has been used as a unit length to measure distances from the body. a^2k_0 has been used as a unit displacement to draw the streamlines and the constant density levels. It may be shown that the slopes of the curves joining the crests and troughs (shown dotted in the figures) are asymptotically $-\cot\xi^*$ so that the approximate rate at which the disturbance spreads outwards is $\frac{1}{2}rW$. A value of $\frac{3}{2}\pi$ for bk_0 was used in the calculations.

9. The wave resistance

The wave resistance to the motion of the body is found from a consideration of the pressures on its surface. If the static thrust is neglected the upthrust is

$$R = 2\pi \int_{-b}^{b} dz\, p \sin\eta \frac{ds}{dz} f(z),$$

as shown in figure 15. p is the perturbed pressure. Euler's equation of motion relative to the body for horizontal motion under steady conditions is

$$W\frac{\partial u}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

when products of small quantities have been dropped. The assumption that $1/\rho \simeq 1/\rho_0$ gives

$$p = W\rho_0 \int_r^\infty \frac{dr}{r}\frac{\partial^2\psi}{\partial z^2} = \frac{W^2\rho_0}{2}\int_0^\infty dk\, k^2F(k)\, H_0^{(1)}(r\Lambda) \exp\left[i(\operatorname{sgn} W)\, kz\right].$$

Further, we assume that the body is slender so that $ds/dz \simeq 1$ and $\sin\eta \simeq f'(z)$. If we write $\rho_0(z) \simeq \rho_0(1-\beta z)$, where $\rho_0 =$ density at $z = 0$, then we obtain

$$\frac{R}{\pi\rho_0 W^2} = \operatorname{Im}\int_{-b}^{b} dz(1-\beta z) f(z) f'(z) \int_0^\infty dk\, k^2 F(k) \exp\left[i(\operatorname{sgn} W)\, kz\right] H_0^{(1)}(r_a\Lambda).$$

Since $f(z)$ is even, this becomes

$$\begin{aligned} \frac{R}{2\pi\rho_0 W^2} = (\operatorname{sgn} W)\int_0^{k_0} dk\, k^2F(k)\int_0^b dz\, f'(z) f(z) \sin kz\, J_0(r_a\lambda_1) \\ -\beta \operatorname{Im}\int_0^\infty dk\, k^2 F(k)\int_0^b dz\, f'(z) f(z)\cos kz\, H_0^{(1)}(r_a\Lambda). \end{aligned}$$

The latter integral represents the effect of the change in virtual mass, and is neglected because it is multiplied by β. If $r_a\lambda_1$ is small over a large part of the range $0 < k < k_0$, so that $J_0(r_a\lambda_1) \simeq 1$, then approximately

$$R = -(\operatorname{sgn} W)\,\pi\rho_0\,W^2\int_0^{k_0} dk\,k^3\{F(k)\}^2,$$

since $$\int_0^b dz f'(z)f(z)\sin kz = -\frac{k}{2}\int_0^b dz\cos kz\,\{f(z)\}^2 = -\frac{k}{2}F(k).$$

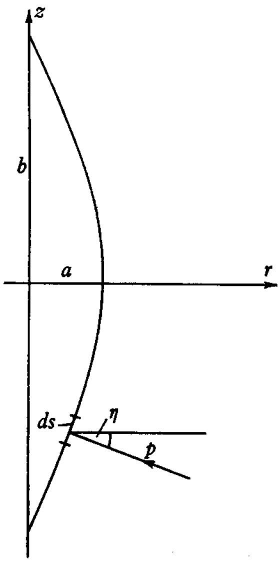

FIGURE 15. Calculation of the wave resistance.

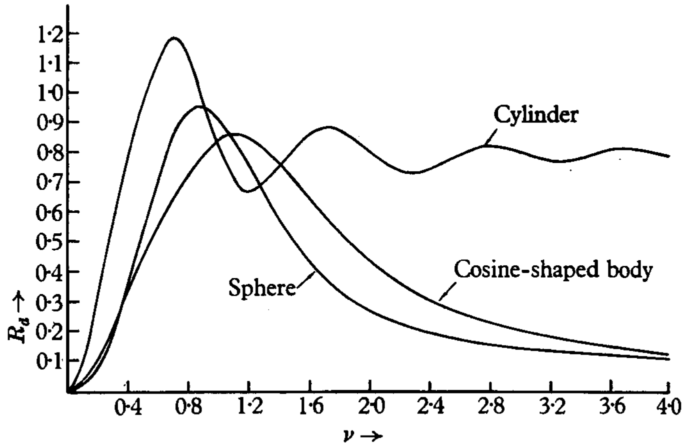

FIGURE 16. Curves of R_d against ν.

$$R_d = \frac{\text{Wave resistance}}{\beta a^4 g\rho_0},$$

$$\nu = \frac{b\sqrt{(g\beta)}}{\pi W}.$$

If the shape of the body is written as $r_a = af_1(z/b)$, where a = half-breadth and b = half-width of the body, the wave resistance is given by

$$\frac{R}{\rho_0 a^4 g\beta} = \frac{1}{\pi\nu^2}\int_0^{\pi\nu} dk\,k^3\{F_1(k)\}^2 = R_d, \quad \text{say,} \tag{19}$$

where $\nu = bk_0/\pi$ and

$$F_1(k) = \int_0^1 dz\cos kz\{f_1(z)\}^2.$$

For small ν, $R_d \propto \nu^2$. In many cases the right-hand side of equation (19) will tend to zero as $\nu \to \infty$, and hence, in general there will be some condition of maximum resistance.

Wave resistance curves for the spindle-shaped body, the sphere and the circular cylinder (whose shape is given by $f(z) = a$, if $|z| < b$, $f(z) = 0$ if $|z| > b$), are shown plotted against ν, that is against a function of velocity, in figure 16. Many assumptions break down for the circular cylinder and its resistance curve is of theoretical interest only.

10. Applications and conclusions

The rise of a thermal through the atmosphere has a velocity given, approximately, by Scorer (1957) $$W = \sqrt{(gBa)},$$

where B = buoyancy = density difference/density, and $2a$ = width of the thermal. To estimate the wave resistance it is assumed that the thermal is a sphere whose diameter is the width of the thermal. The ratio of the wave resistance to the buoyancy is then

$$\frac{\text{resistance}}{\text{buoyancy}} = \frac{3}{4\pi}\frac{a\beta}{B}R_d\left(\sqrt{\frac{a\beta}{\pi^2 B}}\right) = 0\cdot 05\left(\frac{a\beta}{B}\right)^2,$$

roughly, if $a\beta/B < 3$. For the following values of the constants (which may occur in the atmosphere)

$$a = 10^4\,\text{cm.},\quad B = 10^{-2},\quad \beta = 10^{-7}\,\text{cm}^{-1},$$

this ratio is about 0·0005. Thus in these cases the wave resistance to motion is negligible. This remains true over a large range of values of B and β. On the other hand, the following values of the constants:

$$a = 10^5\,\text{cm},\quad B = 3.10^{-3},\quad \beta = 10^{-7}\,\text{cm}^{-1},$$

give a ratio of about 0·5.

As a rough guide, a sphere will experience most wave resistance at a speed given by

$$\frac{a\sqrt{g\beta}}{W} = 3.$$

The ratio of the wave resistance to the weight of the displaced fluid is then about $\frac{1}{4}a\beta$, where a is the radius of the sphere.

Hence in general the dispersion of energy due to wave motion is small. However, when the situation is specially favourable to the surrounding fluid, and the forces involved are small, this dispersion may play a major part in the behaviour of the fluid, as in the case of the larger thermal with a smaller buoyancy (e.g. towards the end of the ascent).

In a similar manner, it may also be possible for these waves to affect the behaviour of turbulent eddies. For instance, if the eddies are large, and if their outer regions move slowly, a considerable proportion of energy may be dispersed by gravity waves. A criterion for this energy disperion to be a maximum would be

$$\frac{a\sqrt{g\beta}}{W} = n,$$

where the number n depends on the shape of the eddy, a is a typical dimension, say the eddy size, and W a velocity typical of its outer regions.

I would like to express my thanks to Dr R. S. Scorer for the very many helpful discussions and guidance he has given me throughout this work.

REFERENCES

Scorer, R. S. 1950 *Proc. Roy. Soc.* A, **201**, 137.

Scorer, R. S. 1957 *J. Fluid Mech.* **2**, 583.

Scorer, R. S. 1958 *Natural Aerodynamics.* London: Pergamon Press.

Watson, G. N. 1944 *Theory of Bessel Functions*, p. 424. 2nd edition. Cambridge University Press.